DOCUMENT RESUME

ED 184 886 SE 030 561

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TITLE Cognitive Models of Algebraic Thought.
INSTITUTION Illinois Univ., Urbana. Computer-Based Education Lab.
PUB DATE Apr 80
NOTE 16p.; Paper presented at the Annual Meeting of the American Educational Research Association (Boston, MA, April 7-11, 1980).

EDRS PRICE MF01/PC01 Plus Postage.
DESCRIPTORS Algebra; *Cognitive Processes; Elementary Secondary Education; Information Processing; *Learning Theories; *Mathematics Education; Problem Solving; Research

ABSTRACT

Human information processing is analyzed in terms of eight fundamental concepts: procedures, operations on procedures, frames, planning language or meta-language, heuristics, knowledge in other forms, critics, and pointers. Examples of this processing are given. (MK)

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Cognitive Models of Algebraic Thought[1]

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I. Fundamental Conceptualization

A. Human information processing, related to the solution of mathematics problems, is analyzed in this presentation in terms of eight fundamental concepts:

1. Procedures: algorithmic step-by-step activities, as (for instance) the cognitive sequence for long division in arithmetic. [cf. Davis, Jockusch, & McKnight, 1978; Davis and McKnight, 1979]

   a. Procedures that depend upon a continuing stream of visual inputs to "jog the memory" - as long division does for most students - are called *visually-moderated sequences (VMS)*.

   b. Procedures that have been learned to the point where they no longer require external inputs to determine the sequence of operations are called *integrated sequences*.

   c. There are also *sub-procedures* and *super-procedures*.

2. *Operations on Procedures.*

   Certain cognitive operations on procedures can be demonstrated: for example, there are "look ahead" and "look backward" operations that allow students, in certain

[1] The research reported here includes results obtained in studies supported by the following grants: NSF SED 77-18047 (Nat'l Science Foundation); NSF SED 79-12740 (Nat'l Science Foundation/Nat'l Institute of Education); NIE G 790141 (Nat'l Institute of Education).

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situations, to predict beforehand what steps will be taken, or to review from memory some of the steps that have been taken. There is evidence of a file record of "procedures used." In certain circumstances a student may "run through" a procedure in his head, and compare the steps to those of some other procedure, perhaps one that he is executing on paper: example: some students can *divide polynomials*

$$x + 2 \overline{)x^4 + 2x^3 - 2x^2 - x + 6}$$

by running through, in their minds, the algorithm for the "long division" of integers

$$31 \overline{)6852}$$

and using the correspondence between the two procedures as a guide.

3. *Frames*

"Frames" is a technical term, introduced by Minsky and others. A *frame* is one form of a "cognitive chunk" [in George Miller's sense], with considerable internal organization. Especially important are the *variables* for which the frame will seek *specific values* from input data, or will alternatively provide *default values*. [Frames are explained in some detail in Davis and McKnight, 1979.]

4. *Planning Language or Meta-Language*

Some students show unmistakable use of a "planning language" that allows them to sketch out a plan of attack *before* they set to work on the details of solving a problem. [cf. especially Davis, Jockusch, and McKnight, 1978].

5. *Heuristics*

A planning language involves descriptors for specific frames or procedures. When an adequate plan CAN be formulated, that can shape subsequent implementation. But more often no adequate plan can be made. Faced with a novel problem for which no complete plan of attack can be formulated, what is a student to do? Most often the student will proceed by a short-term strategy of using "local" cues to produce a "local" response, with no assurance that this will ultimately lead in the right direction.

A better alternative, used by some students, is to carry out a *heuristic analysis* of the problem or task. Heuristic analyses carry no guarantees, but they can *sometimes* be helpful. The list of useful heuristics - at least, those that are used explicitly - includes (for example): try to break a large problem up into pieces, and deal separately with each piece [the art here is to make the separations at suitable points]. What problem does this remind you of? What makes the problem difficult? Can you somehow eliminate that obstacle? Can you make up a similar, simpler problem? [and so on].

6. *Knowledge in Other Forms.*

The subtraction algorithm for, say,

$$\begin{array}{r} 7{,}002 \\ -\ \ 23 \\ \hline \end{array}$$

is, for most students, a "procedure" of a familiar type. (Unfortunately, because of the two zeroes, errors such as the following often creep in:

$$\begin{array}{r} {}^{6}\not{7}{,}0{}^{1}\not{0}2 \\ -\ \ 23 \\ \hline 9 \end{array}$$

...and so on.)

This algorithm *could* be related to other *kinds* of knowledge - for example, money (would you accept 10 one-dollar bills as change for one one-thousand-dollar bill?), Dienes' MAB blocks (is one big block equivalent, base ten, to ten units?), etc. It is an empirical fact that students are surprisingly disinclined to make use of such "other kinds of knowledge" in order to adjust and repair the algorithm.

7. A *critic* is some sort of cognitive entity that passes judgement on processes and results, and helps to identify errors. Thus, for the incorrect subtraction

$$\begin{array}{r} {}^{\downarrow}\not{7},{}^{1}002 \\ -\ \ 23 \\ \hline 5,089 \end{array}$$

a student *should* have (but many do not) a critic based on *size*, that would in effect say: I had $7,000, spent a few dollars, and have $5,000 left. I should have almost $7,000 left. *Something must be wrong!*

8. *Pointers.*

The idea of, say, a *second-degree polynomial* should be closely related to many other ideas. "Pointers" are cognitive objects concerned with establishing such connections. This, *second-degree polynomial* should have pointers to: parabolic graphs: finite difference identification rules; algebraic notation as $y = a_o + a_1x + a_2x^2$; factoring quadratic expressions; the quadratic formula; etc.

II. Examples

A. Building up frames
"Frame-Snatching"

More sophisticated frames are frequently created from simpler frames learned earlier. Given the

principle that discriminations are not finer than they need to be, the earliest frame for addition will in fact be a "binary operation" frame. There is no need, at the earliest stages, to discriminate addition from subtraction, multiplication, etc., because these other operations have not yet appeared. Similarly, when subtraction is first introduced, it is "symmetric subtraction": there is no need for a student to develop a discrimination between a - b and b - a, because the rule will (at this stage) always be: subtract the smaller from the larger. [The language often shows this symmetry explicitly, as in: "the difference between John's age and Mary's age is two years..." - it does NOT say who is older.] Thus, the primary grade "subtraction" frame really deals with "symmetric subtraction", $|x - y|$. Using subtraction items administered by a computer, Davis, McKnight, Parker, and Elrick [1979] were able to show the use of this primary grade "symmetric subtraction" frame by 5th graders who were just beginning to study positive and negative numbers. Thus, the question 3 - 5 = ? would be expected to be answered by: (i) first, the subtraction sign cues the retrieval of the subtraction frame (which is, of course, the "symmetric" subtraction frame); second, the subtraction frame calls for two inputs, receives 3,5, and returns the "answer" 2. In precisely the same way, 7 - ($^-$1) = ? should return the (wrong) answer 6. Again, the question ($^-$3)($^+$5) = ?, for which $^-$15 is the *correct* answer, would be expected to retrieve either the subtraction frame or the addition frame, and thus to return either the answer 8 or else the answer 2, which is what in fact occurred.

This process of retrieving a frame from a single cue (and thus often getting a wrong answer), we have labeled "frame-snatching."

B. *The Friend Phenomenon.* Jamesine Friend reports that, surprisingly, the problem

$$\begin{array}{r} 235 \\ 19 \\ +\ 26 \\ \hline \end{array} \qquad \text{(A)}$$

is harder for students (exhibits morewrong answers) then either

$$\begin{array}{r} 235 \\ 219 \\ +\ 26 \\ \hline \end{array} \qquad \text{(B)}$$

or

$$\begin{array}{r} 235 \\ 69 \\ +\ 21 \\ \hline \end{array} \qquad \text{(C)},$$

despite the fact that one would surely expect (B) and (C) to be harder. One common wrong answer for problem (A) is 480 (the correct answer being 280).

What appears to occur is that when the student gets to the left-most column, the addition frame requests two inputs. In problems (B) and (C) this request is met; in (A), it is *not*. But the frame will not be denied, hence the left-most digit in the lowest row is taken to play the role of the "missing" input. [cf. Davis, McKnight, Parker, and Elrick, 1979].

C. *Karplus's "Recipes"*

Bob Karplus, in some elegant studies of ratio and proportion, reports protocols such as the following:

> [In a story presented to the student, there is a boy John, who is making lemonade with 3 teaspoonfuls of (sweet) sugar and 9 teaspoonfuls of (sour) lemon; a girl, Mary, is also making lemonade, but using 5 teaspoonfuls of (sweet) sugar and 13 teaspoonfuls of (sour) lemon. Appropriate illustrative pictures accompany the story.
>
> The interviewer asks: "Whose lemonade will be sweeter?"]
>
> 1. Student: Let me see. Mary's would be sweeter.

2. Interviewer: Mary's would be sweeter?
Um-hum [thoughtful tone invites further explanation...]

3. Student: Because Mary's has two less lemons in contrast with this [pointing to pictures], with John's.

4. Interviewer: Actually, she has 4 *more*.

5. Student: She *does?* [tone of great surprise]

6. Interviewer: [explaining his preceding remark]
Well, she has 13 compared to 9.

7. Student: Yeah, but in relation to the *sugars*.

8. Interviewer: Could you explain to me how you figured that she has 2 less in relation to sugar?

9. Student: Well, O.K....There's 3 and 9.

10. Interviewer: Uh-huh.

11. Student. And 3 goes into 9 3 times, and then you go 5 and 13,...

12. Interviewer: Yes...

13. Student: 15 goes into 5 3 times[sic!], so it's really too much...

14. Interviewer: Uh-huh. So it's 2 less. And so if Mary wanted to make it come out the same sweetness as John's...

15. Student: It would have to be 15.

16. Interviewer: She'd have to use 15. So, I see she has 2 less... O.K.

This interview excerpt can be split into three sections: utterances 1 through 7 show frame-like behavior, utterances 8-13 show sequential behavior under frame control, and utterances 14-16 are the Interviewer's attempt to re-state the Student's idea in more explicit language.

The student's comparison of the table

	Sugar	Lemon
John	3	9
Mary	5	13

with a different table *which is nowhere in evidence except in her own imagination* is stunning! The alternative table, namely

	Sugar	Lemon
John	3	9
Mary	5	15

is so real to her that she at first rejects the interviewer's "common sense" numbers - *which ARE the numbers that are actually in evidence* - in favor of her "ideal" table.

This, and other evidence in various Karplus interviews, suggests strongly that many students have a "recipe" frame, which allows them great facility in doubling recipes, halving recipes, etc.

D. *The "ropes-and-weight" problem.*

In a not-yet-published study, we audio-taped task-based interviews using the following task:

Each student was to read the following problem, work out an attack on it, and solve it.

> A rope with a ring in one end is looped over two pegs in a horizontal line. The free end, after being passed through the ring, has a weight suspended from it, so that the rope hangs taut. If the rope slips freely over the pegs and through the ring, the weight will descend as far as possible. Assume that the length of the rope is at least four times as great as the distance between the pegs, and that the configuration of the rope is symmetric with respect to the line of the vertical part of the rope. (The symmetry assumption can be justified on the grounds that the rope and weight will take a rest position that minimizes the potential energy of the system.) Find the angle formed at the bottom of the loop. [From Thomas & Finney, *Calculus and Analytic Geometry*, 5th edition (1979), p. 294]

What did those interviews show?

1. Students had far more trouble *getting started* than we had expected.

i) One implication: probably the teacher had been helping set up problems, so that students were not getting enough practice at this crucial stage of the task.

2. The wording in the textbook was far more obscure than we had realized.

   a) Interviews with experienced adults supported the Simon et al. hypothesis that this obscurity was no trouble for experts, because some early cues trigger the retrieval of an appropriate frame, after which the inadequacies of the textbook wording don't matter; guidance comes, not from the statement of the problem, but from the frame.

3. Most students *couldn't even make a correct picture!*

4. Most students could not find the appropriate variables.

5. There was a far greater variation in student response than I had expected. The very *best* students read the problem, immediately and "effortlessly" drew a correct picture, checked for extreme cases to establish that for some θ such that $0 < < \theta < < 180°$ there must be a maximum, explicitly stated a plan of attack, then carried it through to a solution of the problem. *But many students could do almost none of these steps!*

6. One could identify 4 different approaches:

   a) Some students tried to solve the problem using only geometry - clearly doomed to failure, because *physical* law must be involved here somewhere!

   b) Some students looked for a *technique* that would - by "magic," as it were - lead to the solution, e.g., "maybe I can integrate."

c) Some students tried "formula snatching" - often saying, quite explicitly, something like: "I'm sure I must know a formula that will solve this [problem]." Of course they don't.

i) Karplus reports similar results from his observations.

d) Use of the correct *physical principle*. The correct physical principle, of course, is that the configuration will adjust itself so that the potential energy is minimized (or: so that the weight is as near to the floor as possible).

The best students did this quickly and effortlessly. [3 students out of 22]. Most students, by contrast, could not suggest even *one* physical principle - *despite the fact that these students have been together, in the same classes, since the end of sixth grade.* The very idea of a "physical principle" was totally foreign to them.

7. Indeed, the interviews gave more evidence for the already-well-established cognitive principle that *one cannot see something which he is totally unprepared to see.* Most students could not even guess at the relevant physical principle *despite the fact that it is stated, explicitly, TWICE in the statement of the problem itself:* ("...the weight will descend as far as possible..." and "the rope and weight will take a rest position that minimizes the potential energy of the system.")

Although they read these sentences aloud, most of the students *were not prepared to hear what the statements said, and DID NOT HEAR.*

8. Although I think of math (and physics) as a highly creative field, stress heuristic analysis, etc., as I studied these audio-taped interviews I was forced to concede that they raised serious doubts about how far many of these students could progress by "creative" analysis. The interview protocols cry out for simple rote imitation: *most students could solve this problem if (and only if) someone had showed them how to solve it.*

9. There is another question for which I have no satisfactory answer: for those few students who solved the problem almost effortlessly - which, as we have seen, required information that was NOT contained in the problem statement (especially information about the configuration of the pegs) - *where did this other knowledge come from?* Note that *this was not merely random good luck;* the same three students who did the best (and "easiest") job on this are *always* the three who deal with new problems shrewdly and "effortlessly." How come? *(Recall that, with one or two minor exceptions, this entire class of 22 students have been together, in the same class, since the end of grade 6 back in elementary school.)*

SOMEHOW THESE "SUCCESSFUL" STUDENTS SEEM TO HAVE THE NECESSARY COLLECTION OF FRAMES, AND THE MEANS OF RETRIEVING THEM.

Perhaps the most likely explanation is suggested by an analysis of cognitive performance (in a different context) by Greeno [1978]:

> Understanding a sentence is the construction of a representation of the sentence. The implication for teaching students in a way that leads to understanding is that what they have to acquire, in order for understanding to occur, is *a set of knowledge structures and procedures that are used*

1.0
1.1
1.25
1.4
1.6
2.8
2.5
3.2
2.2
3.6
2.0
4.0
1.8
MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

> *in generating [such] representations.* [italics added] [Greeno, 1978, p. 264]

After all, with frames (or other information representations) we always have the possible alternatives of *retrieval from memory* or else *ad hoc construction right then and there.* Perhaps the three highly successful students have a superior capability of constructing appropriate cognitive representations *in their minds,* after which constructing a picture on the paper (or otherwise "understanding" the problem) may indeed by quite straightforward.

E. *The "Greater Than" Relation.*

Observing students, and using appropriate cognitive theory, can give us a new understanding of how students do mathematics. This example shows the value of *procedures.*

Richard J. Shumway, of Ohio State University, has pointed out the discrepancy between formal definitions of

$$a < b \ ,$$

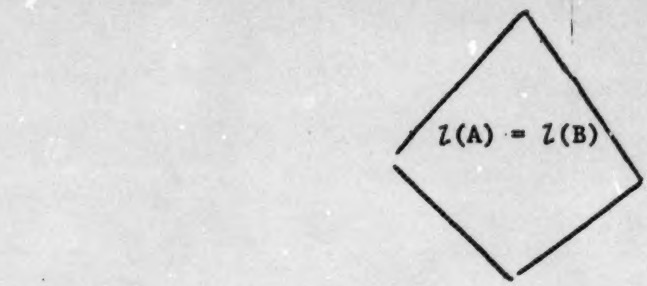
vs. what students actually do to decide whether a < b.

For example, one may define: "a < b if and only if there exists a positive number N such that

$$a + N = b."$$

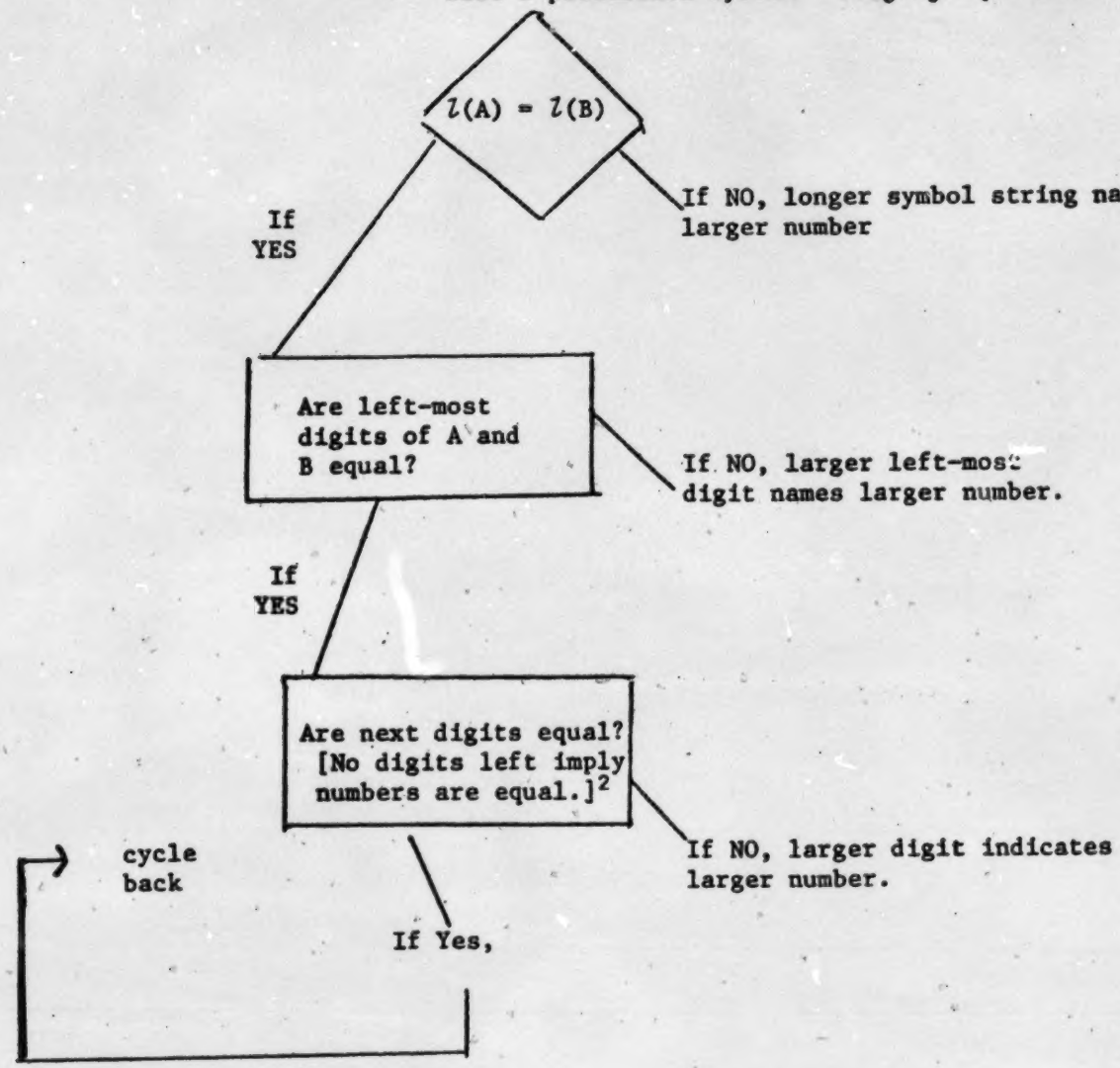
Now, ask a student which is smaller, 31 or 2,986. What thought processes will the student employ?

a) For *positive integers,* the first step in deciding probably looks like this (where $\ell(A)$ denotes the number of digits in the decimal representation of the integer A, etc.)



If NO, longer symbol string names larger number

b) But suppose $\ell(A) = \ell(B)$; what then? Typical students presumably use some step-wise decision procedure - or else a production-system - *roughly* equivalent to this:


ℓ(A) = ℓ(B)
If NO, longer symbol string names larger number
If YES
Are left-most digits of A and B equal?
If NO, larger left-most digit names larger number.
If YES
Are next digits equal? [No digits left imply numbers are equal.]2
If NO, larger digit indicates larger number.
cycle back
If Yes,

[2] In fact, two identical numbers would almost certainly be recognized in the *first* step of the comparison. We have not bothered to indicate this test for identity, which probably occurs at the start of the procedure.

c) This is all very well for positive integers, but something more is needed to deal with negative numbers, rational numbers, etc. For example:

i) Which is larger, -1,239 or -37?

ii) Which is larger, 0.039 or 1.15?

iii) Which is larger, 0.0395 or 0.00953?

I won't try to "flow chart" any hypothetical procedures for these problems.

d) One might well ask how these step-by-step decision procedures relate to an entirely *different kind* of knowledge, dealing either with *money* or with Dienes MAB blocks.

2. The main point of Shumway's example is that older analyses in terms of formal definitions or, cognitively, "having a concept," although these may be valuable for certain purposes, *are NOT as useful for understanding a student's thinking as the "procedure-and-frame" kind of analysis is. Clearly, a student does not find a positive number to add to 0.039 so as to get 1.15, in order to answer the question of whether 1.15 is larger than 0.039.*

III. *Science*

Perhaps the most important aspect of a science is the *list of basic conceptualizations that it employs*. Given different conceptualizations, one asks different questions, and seeks different *kinds* of answers.

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